TAMU Winterschool: Spin in cold atom systems

Ian's outline

I. Meaning of "Spin" in atomic systems

\[(L,S) \rightarrow J, (J+I) \rightarrow F\] coupling

II. Important levels of Alkalai atoms \(nS \rightarrow nD\) transition

III. Structure in ground state

(hyperfine)

IV. Couplings to spin states

- Zeeman \(\rightarrow\) Breit-Rabi equation, RF coupling
- Electric Dipole (Raman)

V. Example (A): creating spin-orbit coupling

- Select two levels (pseudospins), then Raman Couple \(\rightarrow\) Soc
- Introduce RWA, compare RF u.s. Raman (No SOC \(\rightarrow\) SOC)

VI. Interactions: Spinor condensate physics

VII. Example (B): Phase separation in a two component gas.

This lecture doesn't derive the underlying physics, and instead seeks to reveal the effective Hamiltonians which matter for cold atom physics.
Spin in atomic systems, Basics

Today I will consider alkali atoms in their ground $s$ orbital.

In this situation we will need to consider only the nuclear and electronic angular momenta.

$S = \frac{1}{2}$ and $I = \cdots$ (we will see specifics later).

Let’s think about $I$: it is a combination of nuclear spin and orbital angular momentum.

So already we know what I call “spin” isn’t.

The name of the game is to construct a total angular momentum $J \equiv I + S$ (or $L + S + I$), which takes on values $f = i \pm \frac{1}{2}$, e.g., $F = 1$ and $F = 2$ for $^{87}\text{Rb}$. Given this we select some set of interesting pseudospins.

$F = 2$

\[ \begin{array}{c}
1 - \langle \uparrow \rangle \\
1 + \langle \downarrow \rangle
\end{array} \]

or

\[ \begin{array}{c}
1 - \langle \uparrow \rangle \\
1 + \langle \downarrow \rangle
\end{array} \]

In this lecture I will explore some of the single-particle and interaction physics that might go into such a selection.
Hyperfine structure + Zeeman Effect

Here I will consider the properties of atoms in their ground electronic state in a uniform magnetic field.

I just introduced the components of \( L=0 \) atoms which make the spin

\[ \vec{F} = \vec{S} + \vec{I} \].

(Magnetic dipole)

In atoms these are coupled by the Hyperfine interaction

\[ H_{HF} = A(\vec{I} \cdot \vec{S}) \], using the standard trick \( F^2 = (s+I)^2 = S^2 + 2S.I + I^2 \)

or \( \frac{1}{2}(F^2 - S^2 - I^2) = \vec{S} \cdot \vec{I} \)

So

\[ H_{HF} = \frac{A}{2} \left( \vec{F}^2 - \vec{S}^2 - \vec{I}^2 \right) - \frac{A}{2} \left[ f(F+1) - s(s+1) - I(I+1) \right] \]

now in the Alkali atoms \( S = \frac{1}{2}, \) so \( F = I \pm \frac{1}{2}, \) for \( I > 0 \)

\( -\frac{1}{2} \) \( \Delta_HFS = A \left[ I + \frac{1}{2} \right] \)

Now some examples:

<table>
<thead>
<tr>
<th></th>
<th>( ^6\text{Li} )</th>
<th>( ^7\text{Li} )</th>
<th>( ^{23}\text{Na} )</th>
<th>( ^{39}\text{K} )</th>
<th>( ^{40}\text{K} )</th>
<th>( ^{85}\text{Rb} )</th>
<th>( ^{87}\text{Rb} )</th>
<th>( ^{133}\text{Cs} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>1</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{5}{2} )</td>
<td>( \frac{3}{2} )</td>
<td>( \frac{7}{2} )</td>
</tr>
<tr>
<td>( \Delta_HFS )</td>
<td>228 kHz</td>
<td>804 kHz</td>
<td>1.73 GHz</td>
<td>462 kHz</td>
<td>129 GHz</td>
<td>254 kHz</td>
<td>3.03 GHz</td>
<td>6.33 GHz</td>
</tr>
<tr>
<td>( A )</td>
<td>152 kHz</td>
<td>402 kHz</td>
<td>885 kHz</td>
<td>231 kHz</td>
<td>-286 kHz</td>
<td>127 kHz</td>
<td>1.61 GHz</td>
<td>3.42 GHz</td>
</tr>
</tbody>
</table>

So there are a wide range of possible hyperfine splittings which - as we shall see - has important effects.
Hyperfine structure + Zeeman Effect II

Now with the hyperfine effect in the bag we move on to the Zeeman effect.

\[ H_z = \mu_B ( g_S S + g_I I ) \cdot B \]

L sometimes \( \mu_B / \hbar \) depending on units (atomic physicists like \( \hbar \), not \( \tau \)).

So all together we consider

\[ H_z = A (I \cdot S) \cdot \mu_B ( g_S S + g_I I ) \cdot B \]

the solution to this equation is the Breit-Rabi equation.

For \( s=1/2 \), and \( B = B_z \mathbf{e}_z \)

Recall that \( J_z = J_x \pm i J_y \) and \( J_z \downarrow \downarrow \mid j, m_j, m_z \rangle = \hbar \left[ (j \pm m_j)(j+1 \pm m_j) \right]^{1/2} \mid j, m_j \downarrow \downarrow \rangle \)

So \( J_x = \frac{1}{2} (J_+ + J_-) \), \( J_y = \frac{1}{2i} (J_+ - J_-) \) giving \( I \cdot S = I_x S_x + \frac{1}{2} I_y S_y + \frac{1}{2} I_z S_z \).

Thus in the uncoupled basis \( H_z \) couples \( \mid i, m_i, m_z, \pm \rangle \) to \( \mid i, m_i, m_z, \pm \rangle \)

which may be graphically depicted like:

\[
\begin{array}{cccc}
M_F & = & -i - \frac{1}{2} & \ldots & M_F = -i + \frac{1}{2} \\
- & \uparrow & & \uparrow & \\
\mid i, -i; \frac{1}{2} \rangle & \rightarrow & \mid i, i; \frac{1}{2} \rangle & \rightarrow & \mid i, -i; -\frac{1}{2} \rangle
\end{array}
\]

So \( H_z \) decomposes into \( 2 \times 2 \) blocks labeled by \( M_F \) as above.

\[
\frac{1}{\hbar} \begin{pmatrix}
\mu_B B (g_S - g_I) + \frac{t_A}{2} (M_F - \frac{1}{2}) & \frac{t_A}{2} (i + \frac{1}{2} + M_F) \frac{1}{2} (i - \frac{1}{2} - M_F) \\
\frac{t_A}{2} (i + \frac{1}{2} + M_F) \frac{1}{2} (i - \frac{1}{2} - M_F) & -\mu_B B (g_S - g_I) + \frac{t_A}{2} (M_F + \frac{1}{2})
\end{pmatrix}
\]

I will compute the eigenvalues of this matrix on the next page.

\[ M_F \text{ doublets} \rightarrow \left( \mid i, M_F - \frac{1}{2}, \frac{1}{2} \rangle, \mid i, M_F + \frac{1}{2}, -\frac{1}{2} \rangle \right) \]
Hyperfine structure + Zeeman Effect III

To recap, I started with

\[ H_J = A(I_S) + \mu_B (g_S S + g_I I) B \]

and transformed to

\[
\begin{pmatrix}
\frac{\mu_B B (g_S - g_I)}{2} + g_I m_F + \frac{h A}{2} (m_F - \frac{1}{2}) \\
\frac{h A}{2} (l + \frac{1}{2} + m_F) + \left( l + \frac{1}{2} - m_F \right) \frac{1}{2} \\
\frac{h A}{2} (l + \frac{1}{2} - m_F) \frac{1}{2}
\end{pmatrix}
\]

For today eigenvalues are enough:

\[
\frac{E_{J=1}}{h} = \frac{g_I \mu_B B m_F + \frac{h A}{2} \left( \frac{1}{2} \right)}{2} \left( 2 \frac{(2J+1)}{2} + 4 m_F \left( \frac{2 (g_I - g_S) \mu_B B}{A} \right) + \left( \frac{2 (g_I - g_S) \mu_B B}{A} \right)^2 \right) \frac{1}{2}
\]

This is the Breit-Rabi equation, the eigenstates of which are the "spin states" usually used in cold atom systems.

Now let's look at some examples [See Slides]

Keep in mind that ~2kG = 0.2 T is near the upper-field limit for cold atom experiments.

**Clock States**

Depending on one's aims, it can be useful to select a pair of states whose energy difference is 1st order insensitive to external magnetic fields.

In \(^{87}\)Rb near \( B=0 \) there are two such pairs:

\[ |F=1, m_F=0 \rangle, |F=2, m_F=0 \rangle \]

and

\[ |F=1, m_F=-1 \rangle, |F=2, m_F=1 \rangle \]

[See Slides]
RF coupling

Let's focus on just one manifold at small field, then

\[ H = g_F \mu_B F \cdot B, \quad g_F = g_i \pm \frac{g_i - g_j}{2i+1} \]

make \( B = B_0 e_z + \delta B \cos(\omega t) \)

\[ H = g_F \mu_B B_0 \hat{F}_z + g_F \mu_B \cos(\omega t) \; \hat{F} \cdot \delta B \]

now I make a transformation into a frame rotating with angular frequency \( \omega \) and make the RWA
to get: \( H = \Omega \cdot \hat{F} \), where

\[ \Omega = (g_F \mu_B \delta B_y, -g_F \mu_B \delta B_x, g_F \mu_B B_0 - \omega) \]

* only terms \( \perp \) to bias matter
* phase of drive affects \( \hat{F}_x, \hat{F}_y \)
* frequency selects states

\[ \begin{align*}
|1\rangle & \quad \text{with} \quad \hat{F} \quad \text{and} \quad \text{phase} \\
& \quad \text{with} \quad \text{and} \quad \text{phase} \\
\end{align*} \]
Dressed states: RF/Raman

Now we have selected a pair of spin states and we want to couple them.

I. \( \hat{H} = \begin{pmatrix} \hbar k' / \text{cm} & 0 \\ 0 & \hbar k / \text{cm} \end{pmatrix} + \begin{pmatrix} \delta / 2 & \Omega / 2 \\ \Omega / 2 & -\delta / 2 \end{pmatrix} \)

\( \hat{H} = \frac{\hbar k^2}{2m} \mathbb{1} + \frac{\delta}{2} \hat{\sigma}_z + \frac{\Omega}{2} \hat{\sigma}_x \), now some \( \mathbf{U} \) solves this hamiltonian

Since \( \mathbf{U} \) is position independent, it commutes with \( \mathbb{1} \)

so \( \mathbf{U} \hat{H} \mathbf{U}^+ = \frac{\hbar k^2}{2m} \mathbb{1} + \frac{\delta}{2} \hat{\sigma}_z \)

standard

RF-dressed state

* RF evaporation

* RF dressed traps

* Quantum Control

II. Raman dressed \( \vec{\nu} \) "vector light shift" momentum displacement operator \( |k-2k, \uparrow \rangle \langle k| \)

\( \hat{H} = \frac{\hbar k' \vec{\nu}}{2m} \mathbb{1} + \frac{\delta}{2} \hat{\sigma}_z + \frac{\Omega}{2} \begin{pmatrix} 0 & e^{2ikx} \\ e^{-2ikx} & 0 \end{pmatrix} \)

It's most simple to write this out:

\( = \frac{i \hbar k}{2m} \left( k' \langle k, \uparrow \rangle \mathbb{1} \langle k, \uparrow \rangle + \frac{\Omega}{2} \left( |k-2k, \uparrow \rangle \langle k, \uparrow | + H.c. \right) + \frac{\delta}{2} \hat{\sigma}_z \right) \)

now I change my labeling, so \( |k, \uparrow \rangle \rightarrow |k+k_e, \uparrow \rangle \)
\( |k, \downarrow \rangle \rightarrow |k-k_e, \downarrow \rangle \)

\( = \frac{i \hbar k}{2m} \left( e^{i k_e} \langle k, \uparrow \rangle \mathbb{1} \langle k, \uparrow \rangle + \frac{\Omega}{2} \left( |k-k_e, \uparrow \rangle \langle k-\downarrow | + H.c. \right) + \frac{\delta}{2} \hat{\sigma}_z \right) \) change integration
Dressed states: RF/Raman

Now we have selected a pair of spin states and we want to couple them.

\[
\frac{1}{2\pi^2 m} \left( \frac{\hbar^2}{2m} \left( k_{e} \cdot \hat{r} \right)^2 \left< \hat{k}_{e} \cdot \hat{r} \right> \right) \delta \left( k_{e} \cdot \hat{r} \right) + \frac{\hbar}{2} \left( \hat{k}_{e} \cdot \hat{r} \right) + H.\left( \right) + \frac{\hbar}{2} \delta_{2}, \text{ change integration}
\]

\[
= \frac{1}{2\pi} \int \left( \left| q \right>, \left< q \right| \right) \left( \frac{\hbar^2}{2m} \left( q \cdot \hat{k}_{e} \right)^2 \frac{\delta_{2}}{2} + \frac{\hbar}{2} \left( \frac{\delta_{2}}{2} - \delta_{2} \right) \right) \left( \left| q \right>, \left< q \right| \right)
\]

\[
= H(q)
\]

\[
H(q) = \frac{\hbar^2}{2m} \mathbb{1} + \left( \frac{\hbar^2}{2m} \right) \sigma_{z} + \frac{\hbar}{2} \sigma_{z} + \frac{\hbar}{2} \sigma_{x}
\]

Rotation by \( \frac{\pi}{2} \) about \( c_{y} \) (spins only)

\[
H(q) = \frac{\hbar^2}{2m} \mathbb{1} + \frac{\hbar^2}{2m} \sigma_{z} - \frac{\hbar}{2} \sigma_{x} + \frac{\hbar}{2} \sigma_{y}
\]

To Experiment
Interactions

So far my discussion has focused on single particle physics, but the "good stuff" has to do with interacting systems.

For the interatomic $V(r) \sim -C_6/r^6$ potential, it is possible to make a pseudo-potential approximation when

$k \ll \frac{2\pi}{a_{\text{flat}}}$, $a_{\text{flat}} \sim 5\text{nm}$ (generally ok)

Then $V(r) = g \delta(r_{i}-r_{j})$ for a single spin.

Let's consider two atoms with angular momentum $F$ colliding with $B=0$. This implies that angular momentum is conserved.

So $V = \delta(r_{i}-r_{j}) \sum_{g=0}^{2F} g \hat{P}_{g}$, where $j_{a} = \frac{4\pi\hbar^{2}q_{a}}{m}$ and $\hat{P}_{g} = \sum_{m_{a}=-g}^{g} |g,m_{a};l_{a},m_{a}=0\rangle \langle g,m_{a}|$.

For Bosons only even $g$ are allowed and odd for Fermions.

This is the first time spin statistics has come up!

So we have the closure relations for Bosons and Fermions

\[ \begin{array}{ll}
\langle g_{a} = 0 | \hat{P}_{g_{a}} | g_{a} = 0 \rangle & = 1 \quad \text{even} \\
\langle g_{a} = 0 | \hat{P}_{g_{a}} | g_{a} = 0 \rangle & = 0 \quad \text{odd}
\end{array} \]

Bosons\qquad\text{Fermions}

Lastly recall $\hat{F}_{i} \cdot \hat{F}_{2} = \frac{Z^{2}}{2} [\hat{F}_{i} \cdot \hat{F}_{2} - \hat{F}_{2} \cdot \hat{F}_{2}] = \frac{Z^{2}}{2} [g(g+1) - 2f(f+1)]$

Multiplying this by our projector: $\hat{F}_{i} \cdot \hat{F}_{2} = \sum_{g=0}^{2F} \hat{F}_{i} \cdot \hat{F}_{2} \hat{P}_{g} = \sum_{g=0}^{2F} \lambda_{g} \hat{P}_{g}$

At this point it's difficult to go further w/o picking $F$
Interactions: Examples

\[ \lambda_s = \frac{\hbar^2}{2}[g(g+1) - 2f(2f+1)] \]

\[ \hat{V} = \delta(r_1 - r_2) \sum q_a \hat{P}_a \]

F=0 (Bosons, trivial): \[ V = \delta(r_1 - r_2) g_0 \hat{P}_0, \text{ but } 1 = \hat{P}_0 \]

This can be rewritten using \( \hat{F}_i \hat{F}_i = -2 \hat{P}_0 + \hat{P}_2 \)

\[ V = \delta(r_1 - r_2) \left[ C_0 (\hat{P}_0 + \hat{P}_2) + C_2 \hat{F}_i \hat{F}_i \right] \]

with \( C_0 = \frac{g_0 + 2g_2}{3}, \quad C_2 = \frac{2g_2 - g_0}{3} \)

This is pretty stunning:

The most general potential is \[ V = \frac{1}{2} \int d^3 x \ g_{ijkl} \Psi_i(x) \Psi_j^+(x) \Psi_k(x) \Psi_l(x) \]

which has 81 coefficients.

Exchange symmetry \( g_{ijkl} = g_{jikl} \) and time reversal \( g_{ijkl} = g_{klji} \)

reduces this to a still large 21!

where the above gives just 2!

If I look at just the \( C_0 \) term: \[ V_0 = \frac{C_0}{2} \int d^3 x \ (\Psi_i \Psi_i^+) \]

\[ = \frac{C_0}{2} \int d^3 x : (\sum \hat{N}_i)^2 : \text{ density-density} \]
Fermions

$F = \frac{1}{2}$ fermions

This is most simple in second quantized notation

$$V = \frac{1}{2} \int d^3x \left( q^+_i q^+_j q_i q_j + q^+_d q^+_d q^+_d q^+_d + \text{h.c.} \right)$$

So $k = 0$ and $i = j$

$$V = \frac{1}{2} \int d^3x \left( q^+_t q^+_t q_t q_t + q^+_d q^+_d q^+_d q^+_d + \text{h.c.} \right)$$

$$V = \frac{3}{2} \int d^3x \left( n^+_p n_p \right)$$

So this is just a density-density interaction